# HOSSAM GHANEM

## (24) 8.4 *Integrals of Rational Function (B)*

**Example 6** Evaluate $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}dx$ 9 May 1997

### Solution

$x^3-x^2-x+1=(x^3-x^2)-(x-1)=x^2(x-1)-(x-1)$
$=(x-1)(x^2-1)=(x-1)(x-1)(x+1)=(x-1)^2(x+1)$

$$\begin{array}{r|l} x^3-x^2-x+1 & x^4-2x^2+4x+1 \\ \hline & x+1 \\ -x^4+x^3+x^2-x & \\ \hline & +x^3-x^2+3x+1 \\ & -x^3+x^2+x-1 \\ \hline & +4x \end{array}$$

$$\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}=x+1+\frac{4x}{(x-1)^2(x+1)}$$

$$\frac{4x}{(x-1)^2(x+1)}=\frac{A}{(x+1)}+\frac{B}{(x-1)}+\frac{C}{(x-1)^2}$$

$A(x-1)^2+B(x+1)(x-1)+C(x+1)=4x$

at $x=1$ ⇨ $c(2)=4$ ⇨ $\boxed{C=2}$

at $x=-1$ ⇨ $A(4)=-4$ ⇨ $\boxed{A=-1}$

at $x=0$ ⇨ $(-1)(1)+B(1)(-1)+(2)(1)=0$ ⇨ $-B+1=0$ ⇨ $\boxed{B=1}$

$$\therefore \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}=x+1-\frac{1}{(x+1)}+\frac{1}{(x-1)}+\frac{2}{(x-1)^2}$$

$$I=\int\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}dx=\int\left(x+1-\frac{1}{(x+1)}+\frac{1}{(x-1)}+\frac{2}{(x-1)^2}\right)dx$$

$$=\frac{1}{2}x^2+x-\ln|x+1|+\ln|x-1|-\frac{2}{x-1}+c$$

**Example 7** Evaluate $\int \frac{(2\sin x - 3)\cos x}{\sin^2 x + 3\sin x + 2}\,dx$

## Solution

Let $t = \sin x \qquad dt = \cos x\,dx$

$$I = \int \frac{2t-3}{t^2+3t+2}\,dt$$

$$\frac{2t-3}{t^2+3t+2} = \frac{2t-3}{(t+2)(t+1)} = \frac{A}{(t+2)} + \frac{B}{(t+1)}$$

$$A(t+1) + B(t+2) = 2t - 3$$

at $t = -2$ $\Rightarrow A(-1) = 2(-2) - 3$ $\Rightarrow -A = -7$ $\Rightarrow \boxed{A = 7}$

at $t = -1$ $\Rightarrow B(1) = 2(1) - 3$ $\Rightarrow \boxed{B = -1}$

$$I = \int \frac{7}{(t+2)} - \frac{1}{(t+1)}\,dt = 7\ln|t+2| - \ln|t+1| + c = 5\ln(2+\sin x) - \ln(1+\sin x) + c$$

**Example 8** * Evaluate the following integral $\int \frac{3x^2+1}{(x+1)^2(x^2+1)}\,dx$ 52 July 24, 2010

## Solution

$$\frac{3x^2+1}{(x+1)^2(x^2+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+1}$$

$$A(x+1)(x^2+1) + B(x^2+1) + (Cx+D)(x+1)^2 = 3x^2 + 1$$

at $x = -1$ $B(2) = 3(1) + 1$ $2B = 4$ $\boxed{B = 2}$

at $x = 0$ $A(1)(1) + (2)(1) + D(1) = 1$ $A + D = -1 \rightarrow \boxed{1}$

at $x = 1$ $A(2)(2) + (2)(2) + (C+D)(4) = 3 + 1$ $4A + 4D + 4C = 0$

$4(A+D) + 4C = 0$ $-4 + 4C = 0$ $\boxed{C = 1}$

at $x = 2$ $A(3)(5) + (2)(5) + (2+D)(9) = 12 + 1$ $\rightarrow 15A + 10 + 18 + 9D = 13$

$15A + 9D + 28 = 13$ $\rightarrow 15A + 9(-1-A) = -15$

$15A - 9 - 9A = -15$ $6A = -6$ $\boxed{A = -1}$

From $\boxed{1}$ $-1A + D = -1$ $\boxed{D = 0}$

$$I = \int \frac{3x^2+1}{(x+1)^2(x^2+1)}\,dx = \int \left(\frac{-1}{x+1} + \frac{2}{(x+1)^2} + \frac{x}{x^2+1}\right)dx$$

$$= -\ln|x+1| - \frac{2}{x+1} + \frac{1}{2}\ln(x^2+1) + C$$

**Example 9**
40 August 7 , 2011

( 3 pts ) Evaluate the following integral $\int \frac{2x^2 - x + 1}{(x-1)(x^2-1)}\,dx$

## Solution

$$(x-1)(x^2-1) = (x-1)(x-1)(x+1) = (x-1)^2(x+1)$$

$$\frac{2x^2-x+1}{(x-1)(x^2-1)} = \frac{2x^2-x+1}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$A(x-1)^2 + B(x+1)(x-1) + C(x+1) = 2x^2 - x + 1$$

at $x = -1$ $A(4) = 2(1) - (-1) + 1$ $\rightarrow 4A = 4$ $\rightarrow \boxed{A = 1}$

at $x = 1$ $C(2) = 2(1) - (1) + 1$ $\rightarrow 2C = 2$ $\rightarrow \boxed{C = 1}$

at $x = 0$ $(1)(1) + B(1)(-1) + (1)(1) = 1$ $\rightarrow 2 - B = 1$ $\rightarrow \boxed{B = 1}$

$$\int \frac{2x^2-x+1}{(x-1)(x^2-1)}\,dx = \int \left(\frac{1}{x+1} + \frac{1}{x-1} + \frac{1}{(x-1)^2}\right)dx = \ln|x+1| + \ln|x-1| - \frac{1}{x-1} + c$$

# Homework

| | | | |
|---|---|---|---|
| 1 | Evaluate the integral | $\int \frac{1+x^4}{1-x^4}dx$ | |
| 2 | Evaluate the integral | $\int \frac{x^4+x^2+1}{x^3+x}dx$ | |
| 3 | Evaluate the integral | $\int \frac{x^5}{(x^2-4)(x^2+4)}dx$ | 44 July 2007 |
| 4 | Evaluate the integral | $\int \frac{\sin 2x}{2+\cos x}dx$ | 2 May 1995 |
| 5 | Evaluate the integral | $\int \frac{3x^2+5x+10}{(x+2)(x^2-2x+4)}dx$ | 22 December 2000 |
| 6 | Evaluate the integral | $\int \frac{2x^2}{(x^2+1)(x-1)}dx$ | 23 May 2001 |
| 7 | Evaluate the integral | $\int \frac{2-3x}{(x^2+1)(x+1)^2}dx$ | 24 August 2001 |
| 8 | Evaluate the integral | $\int \frac{x^2+x+2}{x(x^2+2x+2)}dx$ | 25 December 2001 |
| 9 | Evaluate the integral | $\int \frac{x^2-4}{x^2+1}dx$ | 27 December 2002 |
| 10 | Evaluate the integral | $\int \frac{2}{(1+x^2)(1+x)}dx$ | 28 May 2003 |
| 11 | Evaluate the integral | $\int \frac{4x}{8-x^3}dx$ | 29 May 2003 |
| 12 | Evaluate the integral | $\int \frac{4x}{x^3+3x^2+3x+9}dx$ | 30 July 2003 |
| 13 | Evaluate the integral | $\int \frac{x^3+1}{(x^2+1)^2}dx$ | 32 December 2003 |
| 14 | Evaluate the integral | $\int \frac{x^2+x+1}{(x^2+2)(x-1)}dx$ | 33 May 2004 |
| 15 | Evaluate the integral | $\int \frac{4x^2-3x+2}{4x^2-4x+3}dx$ | 36 June 2005 |
| 16 | Evaluate the integral | $\int \frac{2x^3-2x^2-3x-5}{(x+1)^2(x^2+2)}dx$ | 39 December 2005 |

# Homework

| | | | |
|---|---|---|---|
| 17 | Evaluate the integral | $\int \frac{7x^2+x+3}{(x+1)(2x^2+1)}dx$ | 46 July 2008 |
| 18 | Evaluate the integral | $\int \frac{x^3+4x}{x^2-4}dx$ | 47 December 2008 |
| 19 | Evaluate the integral | $\int \frac{2-3x^2}{(2x+1)(x^2+2x+2)}dx$ | 48 May 2009 |
| 20 | Evaluate | $\int \frac{x^2+14}{(x-2)(x^2+2x+10)}dx$ | 49 August 2009 |
| 21 | Evaluate the following ( $2\,{}^1/_2$ points) | $\int \frac{x^2+x+1}{x^2-1}dx$ | 56 11 December 2011 |
| 22 | Evaluate the following integral | $\int \frac{x^5}{x^4-16}dx$ | |

**22** Evaluate the following integral $\int \frac{x^5}{x^4-16}dx$

## Solution

$x^4-16=(x^2-4)(x^2+4)=(x-2)(x+2)(x^2+4)$

$$\begin{array}{r|l} x^4-16 & x^5 \\ \hline -x^5+16x & x \\ \hline 16x & \end{array}$$

$\therefore \frac{x^5}{x^4-16}=x+\frac{16x}{x^4-16}$

$\frac{16x}{x^4-16}=\frac{16x}{(x-2)(x+2)(x^2+4)}=\frac{A}{(x-2)}+\frac{B}{(x+2)}+\frac{Cx+D}{(x^2+4)}$

$A(x+2)(x^2+4)+B(x-2)(x^2+4)+(Cx+D)(x-2)(x+2)=16x$

$at\ \ x=-2 \quad \Rightarrow B(-4)(8)=-32 \quad \Rightarrow \boxed{B=1}$

$at\ \ x=2 \quad \Rightarrow A(4)(8)=32 \quad \Rightarrow \boxed{A=1}$

$at\ \ x=0 \quad \Rightarrow (1)(2)(4)+(1)(-2)(4)+D(-2)(2)=0 \quad \Rightarrow 8-8-4D=0 \quad \Rightarrow \boxed{D=0}$

$at\ \ x=1 \quad \Rightarrow (1)(3)(5)+(1)(-1)(5)+C(-1)(3)=16 \quad \Rightarrow 15-5-3C=16$

$10-3C=16 \quad \Rightarrow -3C=6 \quad \Rightarrow \boxed{C=-2}$

$\therefore I=\int \frac{x^5}{x^4-16}dx=\int\left[\frac{1}{x-2}+\frac{1}{x+2}-\frac{2x}{x^2+4}+x\right]dx$

$=\ln|x-2|+\ln|x+2|-\ln|x^2+4|+\frac{1}{2}x^2+c$

**20** Evaluate $\int \frac{x^2+14}{(x-2)(x^2+2x+10)}dx$ 49 August 2009

## Solution

$\frac{x^2+14}{(x-2)(x^2+2x+10)}=\frac{A}{(x-2)}+\frac{Bx+C}{(x^2+2x+10)}$

$A(x^2+2x+10)+(Bx+C)(x-2)=x^2+14$

$at\ \ x=2 \quad \Rightarrow 18A=18 \quad \Rightarrow \boxed{A=1}$

$at\ \ x=0 \quad \Rightarrow 10+(-2)C=14 \quad \Rightarrow 10-2C=14 \quad \Rightarrow -2C=4 \quad \Rightarrow \boxed{C=-2}$

$at\ \ x=1 \quad \Rightarrow (1)(13)+(B-2)(-1)=15 \quad \Rightarrow -(B-2)=2 \quad \Rightarrow -B+2=2 \quad \Rightarrow \boxed{B=0}$

$I=\int \frac{x^2+14}{(x-2)(x^2+2x+10)}dx=\int\left(\frac{1}{(x-2)}-\frac{2}{(x^2+2x+10)}\right)dx$

$=\int\left(\frac{1}{(x-2)}-\frac{2}{(x+1)^2+9}\right)dx=\ln|x-2|-\frac{2}{3}\tan^{-1}\left(\frac{x+1}{3}\right)$

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